GENERALIZATION OF BERG–DIMOVSKI CONVOLUTION IN SPACES OF ANALYTIC FUNCTIONS

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In the space $\mathcal{H}(G)$ of functions analytic in a $p$-convex region $G$ equipped with the topology of compact convergence, we construct a convolution for the operator $J_p + L$, where $J_p$ is the generalized Gel’fond–Leont’ev integration operator and $L$ is a linear continuous functional on $\mathcal{H}(G)$. This convolution is a generalization of the well-known Berg–Dimovski convolution. We describe the commutant of the operator $J_p + L$ in $\mathcal{H}(G)$ and obtain the representation of the coefficient multipliers of expansions of analytic functions in the system of Mittag–Leffler functions.

Classical convolutions play an important role in distribution theory, harmonic analysis, and operational calculus. However, the problem of finding new convolutions is still inadequately studied. Therefore, it is important to develop new methods for the solution of this problem in special functional spaces.

The concept of convolution for a linear operator $T$ acting in a linear space $X$ was introduced by Dimovski (see, e.g., [1]). A bilinear commutative associative operation $*: X \times X \to X$ is called convolution for $T$ on $X$ if

$$T(f * g) = (Tf) * g \quad \forall f, g \in X.$$  

An arbitrary linear operator $T$ that acts in $X$ and satisfies (1) is called a multiplier of convolution $*$. If $X$ is a topological vector space, continuous convolutions are considered.

Berg [2] and Dimovski [3] constructed a convolution in classical functional spaces for the operator $T = J + L$, where $J$ is the operator of ordinary integration and $L$ is a fixed linear continuous functional acting in the corresponding space. This convolution is called the Berg–Dimovski convolution. In [1], some properties of this convolution were studied and, with its help, representations of the coefficient multipliers of Dirichlet expansions in different functional spaces were obtained.

In this paper, we construct a convolution for the operator $T = J_p + L$ in spaces of analytic functions ($J_p$ is the generalized Gel’fond–Leont’ev integration operator [4]). We use the general method for the solution of such problems suggested in [5]. The convolution constructed in the present paper generalizes the classical Berg–Dimovski convolution.

Let $G$ be an arbitrary region of the complex plane star-shaped with respect to the origin 0. Let $\mathcal{H}(G)$ denote the space of all functions analytic in $G$ and equipped with the topology of compact convergence [6], and let $\mathcal{L}(\mathcal{H}(G))$ be the set of all linear continuous operators acting in $\mathcal{H}(G)$. Denote by $\mathcal{H}'(G)$ the space of all linear continuous functionals on $\mathcal{H}(G)$. For a positive constant $\rho$, let $J_p$ denote the generalized Gel’fond–Leont’ev integration operator continuously acting in $\mathcal{H}(G)$ according to the rule

$$(J_pf)(z) = z \left[ \Gamma(1/\rho) \right]^{-1} \int_0^1 (1-t)^{1/\rho-1} f(zt^{1/\rho}) \, dt.$$  

First, we fix an arbitrary functional $L \in \mathcal{H}'(G)$ and describe the commutant of the operator $J_p + L$ in $\mathcal{H}(G)$, i.e., find representations of all operators $T \in \mathcal{L}(\mathcal{H}(G))$ satisfying the equality

$$C \mathcal{T} = \mathcal{T} \mathcal{C}$$

where

$$\mathcal{T}(f) = (J_pf)(z)$$

for $f \in \mathcal{H}(G)$ and $z \in G$. This convolution is a generalization of the Berg–Dimovski convolution and can be used to prove that the commutant of $J_p + L$ in $\mathcal{H}(G)$ is a $\mathcal{H}'(G)$-module.

Consider the Mittag-Leffler function [7]
\[ E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n/\rho + 1)}. \]

Since the system \( \{E_\rho(\lambda z) \mid \lambda \in \mathbb{C} \} \) is complete in \( \mathcal{H}(G) \), each operator \( T \in \mathcal{L}(\mathcal{H}(G)) \) is associated with the characteristic function \( t(\lambda, z) = T[E_\rho(\lambda z)] \), which is entire with respect to \( \lambda \) and analytic with respect to \( z \) in \( G \); furthermore, different operators are associated with different characteristic functions. By analogy, the entire function \( l(\lambda) = L(E_\rho(\lambda z)) \) is called characteristic for the functional \( L \).

Assume that an operator \( T \in \mathcal{L}(\mathcal{H}(G)) \) commutes with \( J_\rho + L \). Acting by the left-hand side and right-hand side of equality (2) on the function \( E_\rho(\lambda z) \), we obtain the following equation for the characteristic function \( t(\lambda, z) \) of the operator \( T \):
\[
t(\lambda, z) - \lambda J_\rho [t(\lambda, z)] = \lambda l_1(\lambda) + (1 - \lambda l(\lambda)) \varphi(z),
\]

where \( l_1(\lambda) = L(t(\lambda, z)) \) and \( \varphi(z) = T1 \) (one should take into account that \( \lambda J_\rho [E_\rho(\lambda z)] = E_\rho(\lambda z) - 1 \)).

Since
\[
(E - \lambda J_\rho)^{-1} = \sum_{k=0}^{\infty} \lambda^k J_\rho^k,
\]
where \( E \) is the identity operator and \( \lambda \in \mathbb{C} \), it follows from (3) that
\[
t(\lambda, z) = (1 - \lambda l(\lambda)) \sum_{k=0}^{\infty} \lambda^k (J_\rho^k \varphi)(z) + \lambda l_1(\lambda) E_\rho(\lambda z);
\]
here, the series on the right-hand side converges in the topology of the space \( \mathcal{H}(G) \) for all \( \lambda \in \mathbb{C} \). Acting by the functional \( L \) on both sides of the last equality, we get
\[
l_1(\lambda) = \sum_{k=0}^{\infty} \lambda^k (J_\rho^k \varphi)(\zeta).
\]
Thus,
\[
t(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k (J_\rho^k \varphi)(z) - L \left[ \lambda E_\rho(\lambda \zeta) \sum_{k=0}^{\infty} \lambda^k (J_\rho^k \varphi)(z) - \lambda E_\rho(\lambda z) \sum_{k=0}^{\infty} \lambda^k (J_\rho^k \varphi)(\zeta) \right]
\]
for \( \lambda \in \mathbb{C} \) and \( z \in G \). Consider the generalized Gel'fond–Leont'ev differentiation operator \( D_\rho \) defined on the elements of the system \( \{E_\rho(\lambda z) \mid \lambda \in \mathbb{C} \} \) complete in \( \mathcal{H}(G) \) by the equality \( D_\rho [E_\rho(\lambda z)] = \lambda E_\rho(\lambda z) \). Below, we use the fact that this operator can be continued to an operator \( D_\rho \) that acts linearly and continuously in \( \mathcal{H}(G) \) [8]. Denote by \( \Lambda \) the functional \( \mathcal{H}'(G) \) acting according to the rule \( \Lambda(f(\zeta)) = f(0) - L((D_\rho f)(\zeta)) \).

For \( \lambda \in \mathbb{C} \) and \( z \in G \), the function \( t(\lambda, z) \) can now be represented in the form
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\[ t(\lambda, z) = \frac{D_\rho A}{z} \left[ t_1(\lambda, z, \zeta) \right], \]  

(4)

where

\[ t_1(\lambda, z, \zeta) = \sum_{k=0}^{\infty} \lambda^k (J_{\rho}^{k+1} \varphi)(z) - \sum_{k=0}^{\infty} \lambda^k (J_{\rho}^{k+1} \varphi)(\zeta). \]  

(5)

Thus, we have proved the following statement:

**Lemma 1.** If an operator \( T \in \mathcal{L}(\mathcal{H}(\mathcal{G})) \) commutes with the operator \( J_\rho + L \), then its characteristic function \( t(\lambda, z) \) can be represented in the form (4), (5), where \( \varphi(z) = T1 \).

For the renewal of the action of the operator \( T \) on an arbitrary function from the space \( \mathcal{H}(\mathcal{G}) \) by using its characteristic function \( t(\lambda, z) \), we transform the right-hand side of (4).

Following [9], we say that a region \( \mathcal{G} \subset \mathbb{C} \) is \( \rho \)-convex if there exists a sequence of \( \rho \)-convex compact sets exhausting the region \( \mathcal{G} \) from within [7, pp. 333–334]. In what follows, without additional notification, we use the notation and the properties of elementary \( \rho \)-convex sets and generalized Borel transformation \( B \) presented in [7].

Denote

\[ g(\tau, \lambda, z, \zeta) = \frac{[E_\rho(\lambda, z) E_\rho(\zeta, \tau) - E_\rho(z, \tau) E_\rho(\lambda, \zeta)]}{(\lambda - \tau)}. \]

Let us study the properties of the generalized Borel transform of this function.

**Lemma 2.** For any \( \rho \)-convex compact set \( M \), the function

\[ h_1(\tau, \lambda, z, \zeta) = \frac{B[g(\tau, \lambda, z, \zeta)]}{\tau} \]

is analytic on the set \( CM \times \mathbb{C} \times \mathbb{M} \times \mathbb{M} \).

**Proof.** Let \( M \) be an arbitrary \( \rho \)-convex compact set. Then

\[ M = \bigcap_{i \in I} D_\rho^*(\theta_i; v_i), \]

where \( I \subset \mathbb{R} \) is a certain family of indices, and \( D_\rho^*(\theta_i; v_i) \) are closed elementary \( \rho \)-convex regions. To prove the analyticity of the function \( h_1(\tau, \lambda, z, \zeta) \) with respect to the variable \( \tau \), it suffices to show that the generalized Borel transform of \( g(\tau) = g(\tau, \lambda, z, \zeta) \) can be analytically extended to the set \( D_\rho(\theta_0; v_0) = \mathbb{C} \setminus D_\rho(\theta_0; v_0) \) for \( z, \zeta \in D_\rho^*(\theta_0; v_0) \) and \( \lambda \in \mathbb{C} \). Let \( h(\theta; g) \) be the indicator function of an entire function \( g(\tau) \) for degree \( \rho \). By direct calculation, one can easily verify that

\[ h(\theta; g) = \max \{|z|^\rho h(\theta + \text{Arg} z; E_\rho); |\zeta|^\rho h(\theta + \text{Arg} \zeta; E_\rho)\}. \]

By using the formula for the indicator of the Mittag–Leffler function (see [7, p. 329]) and taking into account that it is nonnegative and \( z, \zeta \in D_\rho^*(\theta_0; v_0) \), we get \( h(-\theta_0; g) \leq v_0 \), whence \( D_\rho(\theta_0; v_0) \subset D_\rho(\theta_0; h(-\theta_0; g)) \).
It remains to use the fact that, according to Theorem 6.5 in [7], the function \((B g)(\tau)\) can be analytically extended to the set \(D_\rho(\theta_0; h(-\theta_0; g))\). Let us fix \(\tau \in CM\) and \(\lambda \in \mathbb{C}\) and show that \(h_1(z, \zeta) \equiv h_1(\tau, \lambda, z, \zeta)\) is analytic with respect to \(z\) and \(\zeta\) on \(\hat{M} \times \hat{M}\). It follows from the proof of the first part of the lemma and Theorem 6.5 in [7] that there exists a set \(D_\rho^*(\theta_0; \nu_0)\) such that \(M \subset D_\rho^*(\theta_0; \nu_0)\), \(\tau \in D_\rho(\theta_0; \nu_0)\), and

\[
h_1(z, \zeta) = \rho(e^{-i\theta_0} \zeta)^{\rho-1} \int_0^\infty g(t e^{-i\theta_0}, \lambda, z, \zeta) \exp \left(-t^\rho (e^{-i\theta_0} \zeta)^\rho\right) t^{\rho-1} dt
\]

for \(z, \zeta \in \hat{M}\). It remains to check that integral (6) converges uniformly in \(z\) and \(\zeta\) on an arbitrary compact subset of the form \(K \times K\), where \(K \subset \hat{M}\). For a fixed compact set \(K \subset \hat{M}\), we take \(\varepsilon > 0\) and \(\eta > 0\) so small that

\[
\varepsilon \max_{z \in K} |z|^\rho + \nu_0 + \eta < \Re\left(e^{-i\theta_0} \zeta\right)\rho \quad \text{(this is possible because } \tau \in D_\rho(\theta_0; \nu_0)\text{).}
\]

It follows from the uniform estimate of the modulus of the Mittag-Leffler function in terms of its indicator \(h(\theta)\) presented in [7, p. 228] that one can find a constant \(C > 0\) such that

\[
|E_\rho(z)| \leq C \exp \left[(h(\arg z) + \varepsilon)|z|^\rho\right]
\]

for \(z \in \mathbb{C}\). Since \(K \subset \hat{M} \subset D_\rho^*(\theta_0; \nu_0)\), we have \(|z|^\rho h(\arg z - \theta_0) < \nu_0\) for \(z \in K\). Therefore,

\[
|g(t e^{-i\theta_0}, \lambda, z, \zeta) \exp \left(-t^\rho (e^{-i\theta_0} \zeta)^\rho\right) t^{\rho-1}| \leq 2C \max_{z \in K} |E_\rho(\lambda z)| \exp \left(-\eta t^\rho\right) t^{\rho-1}
\]

for \(|t| \geq |\lambda| + 1\) and \(z, \zeta \in K\). Hence, (6) converges uniformly on \(K \times K\).

Finally, it is easy to see that \(h_1(\tau, \lambda, z, \zeta)\) is entire with respect to \(\lambda\) and belongs to the class \([\rho, \infty)\).

**Lemma 3.** For \(\lambda, \mu, z \in \mathbb{C} \ (\lambda \neq \mu)\), the following equality holds:

\[
\sum_{k=0}^{\infty} \lambda^k \int \frac{E_\rho(\mu z) - E_\rho(\lambda z)}{\lambda - \mu} \text{d}z = \frac{E_\rho(\mu z) - E_\rho(\lambda z)}{\lambda - \mu}.
\]

**Proof.** For \(k \geq 0, \mu \neq 0, \) and \(z \in \mathbb{C}\), we have

\[
J_{z}^{k+1}(E_\rho(\mu z)) = \left(E_\rho(\mu z) - \sum_{m=0}^{k} \mu^m z^m \Gamma(m/\rho + 1)\right) z^{-k-1} = \Delta^{k+1}_\mu(E_\rho(\mu z)),
\]

where \(\Delta\) is the Pommier operator acting according to the rule \((\Delta f)(z) = (f(z) - f(0))/z\). Representing the operator \(\Delta^{k+1}\) in the integral form [6], we get

\[
J_{z}^{k+1}(E_\rho(\mu z)) = \frac{1}{2\pi i} \int_{|\zeta| = R} \frac{E_\rho(\zeta z)\zeta^{-k-1}}{\zeta - \mu} d\zeta, \quad R > |\mu|.
\]
For \( R > \max \{ |\lambda|, |\mu| \} \), we have

\[
\sum_{k=0}^{\infty} \lambda^k \int_{\rho}^{k+1} (E_p(\mu z)) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{E_p(\zeta z)}{(\zeta - \lambda)(\zeta - \mu)} d\zeta.
\]

By calculating the integral, we arrive at the required relation.

**Lemma 4.** Let \( G \) be a \( p \)-convex region in \( \mathbb{C} \) and let \( \varphi \in \mathcal{H}(G) \). Then for any \( p \)-convex compact set \( M \subset G, \lambda \in \mathbb{C}, \) and \( z, \zeta \in M \), function (5) can be represented in the form

\[
t_1(\lambda, z, \zeta) = \frac{1}{(2\pi i)^2} \int_{\gamma} \varphi(\tau) E_p(\lambda \tau) B B[g(\tau, t, z, \zeta)] dt d\tau.
\]

where \( \gamma \) is a closed rectifiable Jordan curve which lies in \( G \) and encloses the compact set \( M \).

**Proof.** By virtue of Lemma 2 and Theorem 6.3 in [7] about the renewal of an entire function by using its generalized Borel transform, for \( \lambda, \mu \in \mathbb{C} \) and \( z, \zeta \in M \), we get

\[
g(\mu, \lambda, z, \zeta) = \frac{1}{2\pi i} \int_{\gamma} E_p(\mu \tau) B B[g(\tau, \lambda, z, \zeta)] d\tau.
\]

On the other hand, by applying (6) twice, we establish that

\[
E_p(\lambda \zeta) \sum_{k=0}^{\infty} \lambda^k \int_{\rho}^{k+1} (E_p(\mu z)) - E_p(\lambda z) \sum_{k=0}^{\infty} \lambda^k \int_{\rho}^{k+1} (E_p(\mu \zeta)) = g(\mu, \lambda, z, \zeta)
\]

for \( \lambda, \mu, z, \zeta \in \mathbb{C} \). Thus, the functions \( \varphi(z) = E_p(\mu z), \mu \in \mathbb{C} \), satisfy the equality

\[
t_1(\lambda, z, \zeta) = \frac{1}{2\pi i} \int_{\gamma} \varphi(\tau) B B[g(\tau, \lambda, z, \zeta)] d\tau.
\]

Since the system \( \{ E_p(\mu z) | \mu \in \mathbb{C} \} \) is complete in \( \mathcal{H}(G) \), relation (9) holds for any \( \varphi \in \mathcal{H}(G) \). Since the function \( g(\tau, \lambda, z, \zeta) \) is symmetric in the first two variables, it can be represented by an expression of type (8) with respect to the second variable. Then equality (9) takes the form (7), and Lemma 4 is proved.

**Lemma 5.** For any \( p \)-convex compact set \( M \), the function \( B B[g(\tau, t, z, \zeta)] \) is analytic on the set \( \mathcal{C} M \times \mathcal{C} M \times \mathcal{M} \times \mathcal{M} \).

The proof of Lemma 5 is analogous to that of Lemma 2.

Let \( \Lambda \in \mathcal{H}'(G) \). Then the functional \( \Lambda \) can be represented in the integral form, namely,

\[
\forall f \in \mathcal{H}(G): \Lambda(f) = \frac{1}{2\pi i} \int_{\gamma} f(z) B B[g(z, t, z, \zeta)] dz.
\]
where \( \gamma(z) \) is the Köthe characteristic function for \( \Lambda \) locally analytic on \( CG \) and such that \( \gamma(\infty) = 0 \), and \( \Gamma \) is a closed rectifiable Jordan curve lying in \( G \) and enclosing all singular points of \( \gamma(z) \) [6]. We say that a compact set \( K \subset G \) is \( \Lambda \)-admissible if all singular points of its Köthe characteristic function lie inside \( K \). Then we can take a contour \( \Gamma \) in a \( \Lambda \)-admissible compact set \( K \) as \( \Gamma \) in (10). Note that if some sequence of compact sets exhausts the region \( G \) from within, then, beginning with a certain number, all these compact sets are \( \Lambda \)-admissible.

**Theorem 1.** Let \( G \) be a \( \rho \)-convex region of the complex plane and let \( L \in \mathcal{H}'(G) \). For an operator \( T \in \mathcal{L}(\mathcal{H}(G)) \) to commute with \( J_\rho + L \), it is necessary and sufficient that, for any \( \Lambda \)-admissible \( \rho \)-convex compact set \( M \subset G \), where \( \Lambda(f) = f(0) - L(D_\rho f) \), a function \( f \in \mathcal{H}(G) \), and \( z \in M \), the following equality hold:

\[
(Tf)(z) = D_\rho \Lambda \left[ \left( \frac{1}{2\pi i} \right)^2 \int \int f(t) \varphi(\tau) B B^*[g(t, \tau, z, \zeta)] d\tau dt \right],
\]

(11)

where \( \gamma \) is a contour lying in \( G \) and enclosing the compact set \( M \), and \( \varphi \) is a function from \( \mathcal{H}(G) \) such that \( \varphi = T1 \).

**Proof.** Necessity. Assume that \( T \) belongs to \( \mathcal{L}(\mathcal{H}(G)) \) and commutes with the operator \( J_\rho + L \). Then, according to Lemmas 1 and 4, for \( \lambda \in \mathbb{C} \) and \( z \in M \), the characteristic function \( \iota(\lambda, z) \) of the operator \( T \) can be represented in the form

\[
\iota(\lambda, z) = D_\rho \Lambda \left[ \iota_1(\lambda, z, \zeta) \right],
\]

where \( \iota_1(\lambda, z, \zeta) \) is given by (7). Consider an operator \( T_1 \) whose action on an arbitrary function \( f \in \mathcal{H}(G) \) with \( z \in M \) is determined by the right-hand side of equality (11). By using Lemma 5, one can easily verify that \( T_1 \in \mathcal{L}(\mathcal{H}(G)) \). Since the characteristic functions of the operators \( T \) and \( T_1 \) coincide, we have \( T = T_1 \) and, hence, \( T \) can be represented in the form (11).

Sufficiency. Consider the operator \( T \) defined by (11). In this case, \( T \in \mathcal{L}(\mathcal{H}(G)) \) and its characteristic function \( \iota(\lambda, z) \) can be represented in the form (4). It follows from (4) that

\[
L[\iota(\lambda, z)] = L \left[ \sum_{k=0}^{\infty} \lambda^k (J_\rho^k \varphi)(z) \right].
\]

Taking this into account, one can easily verify that the equality \( [T(J_\rho + L)](f) = [(J_\rho + L)T](f) \) holds for functions of the form \( f(z) = E_\rho(\lambda z) \), \( \lambda \in \mathbb{C} \), and, hence, for any function \( f \in \mathcal{H}(G) \).

Now let us construct a convolution for the operator \( J_\rho + L \) in \( \mathcal{H}(G) \).

**Theorem 2.** Let \( G \) be a \( \rho \)-convex region of the complex plane, let \( L \in \mathcal{H}'(G) \), and let \( \Lambda(f) = f(0) - L(D_\rho f) \). For any \( \Lambda \)-admissible \( \rho \)-convex compact set \( M \subset G \), functions \( f \) and \( \varphi \) from \( \mathcal{H}(G) \), and \( z \in M \), the relation

\[
\text{...}
\]
defines a continuous convolution for the operator $J_p + L$; here, $\gamma$ is a closed rectifiable Jordan curve lying in $G$ and enclosing $M$.

**Proof.** The fact that the operation $*$ is defined on $H(G) \times H(G)$ and takes values in $H(G)$ follows from the proof of Theorem 1. The fact that the operation $*$ is bilinear, commutative, and continuous is evident. To prove the associativity of the convolution $*$ it suffices to check the validity of the equality $(f \ast g) \ast h = f \ast (g \ast h)$ for functions from the system $\{E^{(\lambda)}(z) | \lambda \in \mathbb{C}\}$, where $E^{(\lambda)}(z) = E_p(\lambda z)$, which is complete in $H(G)$. For $\lambda, \mu, v \in \mathbb{C}$, $\lambda \neq \mu$, we have

$$(E^{(\lambda)} \ast E^{(\mu)}) \ast E^{(v)}(z) = \Lambda_{\zeta} \left[ \frac{E^{(\mu)}(\zeta) E^{(\mu)}(\zeta) - E^{(\mu)}(\zeta) E^{(\lambda)}(\zeta)}{\lambda - \mu} \right].$$

Therefore, for different complex numbers $\lambda, \mu$, and $v$,

$$(E^{(\lambda)} \ast E^{(\mu)}) \ast E^{(v)} = (E^{(\lambda)} \ast (E^{(\mu)} \ast E^{(v)})$$

and, hence, the operation $*$ is a convolution.

Since $E^{(0)}(z) = 1$, we have

$$\mu \in \mathbb{C} \setminus \{0\}; \quad \{1\} \ast E^{(\mu)}(z) = \Lambda_{\zeta} \left[ \frac{E^{(\mu)}(\zeta) - E^{(\mu)}(\zeta)}{\mu} \right] = [(J_p + L) E^{(\mu)}](z).$$

Hence, the equality $(J_p + L)f = \{1\} \ast f$ holds for any function $f \in H(G)$. Thus, for $f, g \in H(G)$,

$$(J_p + L)(f \ast g) = (1 \ast f) \ast g = ((J_p + L)f) \ast g,$$

i.e., the operator $J_p + L$ is a multiplier of the convolution $*$. Theorem 2 is proved.

**Remark 1.** It follows from the proof of Theorem 2 that relation (12) defines a continuous convolution in $H(G)$ for any fixed functional $\Lambda \in H'(G)$ (not only for that generated by the functional $L$).

Convolution (12) constructed above is a generalization of the classical Berg–Dimovski convolution. Indeed, let $p = 1$. In this case, $E_p(z) = \exp(z)$, $D_p = d/dz$, the notion of $\rho$-convexity of the region $G$ is equivalent to the ordinary convexity, and the generalized Borel transformation coincides with the Borel transformation. For $\tau, t, z, \zeta \in \mathbb{C}$, we have

$$g(\tau, t, z, \zeta) = \frac{\exp(iz + \tau \zeta) - \exp(iz + r \zeta)}{r - \tau} = \int_{\zeta}^{\bar{\zeta}} \exp(i \sigma) \exp[\tau(z + \zeta - \sigma)] d\sigma,$$
where the integration is carried out over the segment that connects the points \( z \) and \( \zeta \). Then, for any convex compact set \( M \), the following equality holds for \( \tau, t \in CM \) and \( z, \zeta \in M^* \):

\[
BB \left[ g(\tau, t, z, \zeta) \right] = -\int_{z}^{\zeta} \frac{1}{t - \sigma} \frac{1}{\tau - (z + \zeta - \sigma)} \, d\sigma.
\]

By calculating the integral on the right-hand side of (12), we establish that the relation

\[
(f * \varphi)(z) = -\Lambda \left[ \int_{z}^{\zeta} f(\sigma) \varphi(z + \zeta - \sigma) \, d\sigma \right]
\]

defines a convolution in \( \mathcal{H}(G) \), where \( G \) is a convex region in \( \mathbb{C} \). But (13) is nothing else but the Berg-Dimovski convolution [1].

**Remark 2.** Under the conditions of Theorem 2, convolution (12) has no annihilators in \( \mathcal{H}(G) \). Indeed, assume that there is a function \( f \in \mathcal{H}(G) \) for which \( f * \varphi = 0 \) for all \( \varphi \in \mathcal{H}(G) \). This means that \( f * 1 = 0 \) and, hence, \( ((J_\varphi + L)f)(z) = 0, \ z \in G \). Acting by the operator \( D_\varphi \) on both sides of this equality, we get \( f(z) = 0, \ z \in G \).

With the use of Theorems 1 and 2, the commutant of the operator \( J_\varphi + L \) can be described in another form.

**Corollary 1.** Let \( G \) be a \( \varphi \)-convex region in \( \mathbb{C} \) and let \( L \in \mathcal{H}'(G) \). In order that an operator \( T \in \mathcal{L}(\mathcal{H}(G)) \) commute with the operator \( J_\varphi + L \), it is necessary and sufficient that it be representable in the form

\[
Tf = D_\varphi(f * \varphi), \ f \in \mathcal{H}(G),
\]

where \( * \) is convolution (12) and \( \varphi(z) \) is some function from \( \mathcal{H}(G) \) for which \( \varphi = T 1 \).

**Corollary 2.** If \( G \) is a \( \varphi \)-convex region in \( \mathbb{C} \) and \( L \in \mathcal{H}'(G) \), the commutant of the operator \( J_\varphi + L \) in \( \mathcal{H}(G) \) consists of operators \( T \) that can be represented in the form

\[
Tf = \mu f + f * \psi,
\]

where \( \mu \in \mathbb{C} \) and \( \psi \in \mathcal{H}(G) \).

The proof is based on Corollary 1 and the relation \( D_\varphi(f * \varphi) = \Lambda(\varphi)f + f * (D_\varphi \varphi) \), where \( f, \varphi \in \mathcal{H}(G) \). This relation can easily be verified on functions of the system \( \{E_\varphi(\lambda z) \mid \lambda \in \mathbb{C} \} \) complete in \( \mathcal{H}(G) \).

**Corollary 3.** For \( \varphi \in \mathcal{H}(G) \), \( \lambda \in \mathbb{C} \), and \( z \in G \), the following equality holds:

\[
(\varphi * E^{(\lambda)})(z) = \Lambda \left[ t_1(\lambda, z, \zeta) \right],
\]

where \( t_1(\lambda, z, \zeta) \) is given by relation (5).
In fact, the relation in Corollary 3 has been established in the proof of Theorems 1 and 2.

Let us apply convolution (12) to the description of the coefficient multipliers of expansions of analytic functions in the Mittag-Leffler system. As shown in [8], for any \( p \)-convex compact set \( \overline{D} \) whose interior contains the origin of coordinates, there exists a sequence of complex numbers \( \{ \lambda_n \mid n \in \mathbb{N} \} \) such that each function \( f(z) \) analytic in the region \( G, \overline{D} \subset G, \) can be expanded into the series

\[
f(z) = \sum_{n=1}^{\infty} a_n(f) E_p(\lambda_n z)
\]

which uniformly converges to \( f(z) \) inside \( D \). The coefficients \( a_n(f) \) in representation (14) are given by the relations

\[
a_n(f) = \frac{1}{2\pi i} \int_{\Gamma} f(t) \psi_n(t) dt,
\]

where \( \{ \psi_n(t) \mid n \in \mathbb{N} \} \) is a system of functions biorthogonal to the family \( \{ E_p(\lambda_n z) \mid n \in \mathbb{N} \} \), and \( \Gamma \) is a closed contour lying in \( G \) and enclosing \( \overline{D} \) [8].

Let \( v(\lambda) \) be an entire function from the class \( [p, \infty) \) such that \( v(0) = 1 \). Assume that the indicator of the function \( v(\lambda) \) takes strictly positive values and \( v(\lambda) \) has countably many simple zeros \( \{ \lambda_n \mid n \in \mathbb{N} \} \). Denote by \( \overline{D} \) the \( p \)-convex hull of the set of singular points of the generalized Borel transform of \( v(\lambda) \). Under these conditions, the system \( \{ E_p(\lambda_n z) \mid n \in \mathbb{N} \} \) has a biorthogonal system \( \{ \psi_n(t) \mid n \in \mathbb{N} \} \) [8]. Let \( G \) be some \( p \)-convex region containing \( \overline{D} \). Each function \( f \in \mathcal{H}(G) \) can be associated with the series

\[
f(z) = \sum_{n=1}^{\infty} a_n(f) E_p(\lambda_n z)
\]

whose coefficients are given by (15). Series (15) is called the formal expansion of a function \( f(z) \) in the Mittag-Leffler system of functions \( \{ E_p(\lambda_n z) \mid n \in \mathbb{N} \} \). Following [1], we say that an operator \( T \in \mathcal{L}(\mathcal{H}(G)) \) is a coefficient multiplier of expansion (16) if one can find a complex sequence \( \{ \mu_n \mid n \in \mathbb{N} \} \) such that \( a_n(Tf) = \mu_n a_n(f) \) for all \( n \geq 1 \) and \( f \in \mathcal{H}(G) \); in this case, the sequence \( \{ \mu_n \mid n \in \mathbb{N} \} \) is called a multiplier sequence of expansion (16).

First, we present another method for the calculation of the coefficients \( a_n(f) \). Note that the relation

\[
F(f) = \frac{1}{2\pi i} \int_{\Gamma} \gamma(z) f(z) dz,
\]

where \( \gamma(z) \) is the generalized Borel transform of a function \( v(\lambda) \), and \( \Gamma \) is a closed contour lying in \( G \) and enclosing \( \overline{D} \), defines a functional \( F \in \mathcal{H}'(G) \) whose characteristic function is \( v(\lambda) \).

Lemma 6. For any function \( f \in \mathcal{H}(G) \), coefficients (15) can be calculated by the relation

\[
a_n(f) = -\frac{1}{v'(\lambda_n)} F \left[ \sum_{k=0}^{\infty} \lambda_n^k (f^{[k+1]})(z) \right].
\]
Proof. We fix $n \in \mathbb{N}$. Functional (15) is linear and continuous on $\mathcal{H}(G)$. It is easy to see that the right-hand side of (17) defines a functional $b_n(f) \in \mathcal{H}'(G)$. It follows from the properties of the biorthogonal system $\{ \varphi_n(t) | n \in \mathbb{N} \}$ that $a_n(E_p(\lambda z)) = \varphi(\lambda) / ((\lambda - \lambda_n)\varphi'(\lambda_n))$ for $\lambda \in \mathbb{C}$ [8]. By using Lemma 3, we get

$$b_n(E_p(\lambda z)) = \varphi(\lambda) / ((\lambda - \lambda_n)\varphi'(\lambda_n)).$$

The characteristic functions of the functionals $a_n$ and $b_n$ coincide and, hence, $a_n(f) = b_n(f)$ for any function $f \in \mathcal{H}(G)$.

Consider the functional $L \in \mathcal{H}'(G)$ defined as follows: $L(f) = -F(J_0 f)$, $f \in \mathcal{H}(G)$. In this case, the functional $\Lambda$ in Theorems 1 and 2 that corresponds to $L$ coincides with $F$. Therefore, relation (12) with $\Lambda \equiv F$ defines a continuous convolution for $J_0 + L$ in $\mathcal{H}(G)$.

**Corollary 4.** For $f \in \mathcal{H}(G)$, $z \in G$, and $n \in \mathbb{N}$, the following equalities hold:

$$(f \ast E^{(\lambda_n)})(z) = \varphi'(\lambda_n) a_n(f) E^{(\lambda_n)}(z).$$

This statement is an immediate consequence of Lemma 6 and Corollary 3.

**Theorem 3.** For an operator $T \in L(\mathcal{H}(G))$ to be a coefficient multiplier of expansion (16), it is necessary and sufficient that this operator be representable in the form $Tf = \mu f + \psi \ast f$, where $\mu \in \mathbb{C}$ and $\psi \in \mathcal{H}(G)$.

**Proof.** By virtue of Theorem 2.2.1 in [8], the system of functionals $\{ a_n(f) | n \in \mathbb{N} \}$ possesses the property of uniqueness. Therefore, it follows from Theorem 3.4 in [10] and Corollary 4 that the set of coefficient multipliers of expansion (16) coincides with the set of multipliers of convolution (12). For the operator $J_0 + L$, which is a multiplier of convolution (12), the function $\varphi(z) \equiv 1$ is a cyclic element in the space $\mathcal{H}(G)$. Since $\mathcal{H}(G)$ is a Fréchet space and convolution (12) has no annihilators in $\mathcal{H}(G)$, by virtue of Theorem 1.3.11 in [1], the set of multipliers of convolution (12) coincides with the commutant of the operator $J_0 + L$. To complete the proof it remains to use Corollary 2.

**Corollary 5.** A sequence $\{ \mu_n | n \in \mathbb{N} \}$ is a multiplier sequence of expansion (16) if and only if $\mu_n = \mu - \varphi'(\lambda_n) a_n(\psi)$, $n \in \mathbb{N}$, where $\mu$ is some constant and $\psi \in \mathcal{H}(G)$.

All results of this paper can easily be extended to the case of the space $\mathcal{H}(\overline{G})$.

**REFERENCES**


